

# COVARIANT LIE DERIVATIVES AND FRÖLICHER-NIJENHUIS BRACKET ON LIE ALGEBROIDS

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**ABSTRACT.** We define covariant Lie derivatives acting on vector-valued forms on Lie algebroids and study their properties. This allows us to obtain a concise formula for the Frölicher-Nijenhuis bracket on Lie algebroids.

## 1. INTRODUCTION

The Frölicher-Nijenhuis calculus was developed in the seminal article [2] and extended to Lie algebroids in [10]. It has proven to be an indispensable tool of Differential Geometry. Indeed, different kinds of curvatures and obstructions to integrability are computed by the Frölicher-Nijenhuis bracket. For example, if  $J : TM \rightarrow TM$  is an almost-complex structure, then  $J$  is complex structure if and only if the Nijenhuis tensor  $\mathcal{N}_J = \frac{1}{2}[J, J]_{FN}$  vanishes (this is the celebrated Newlander-Nirenberg theorem [9]). If  $F : TM \rightarrow TM$  is a fibrewise diagonalizable endomorphism with real eigenvalues and of constant multiplicity, then the eigenspaces of  $F$  are integrable if and only if  $[F, F]_{FN} = 0$  (see [4]). Further, if  $P : TE \rightarrow TE$  is a projection operator on the tangent spaces of a fibre bundle  $E \rightarrow B$ , then  $[P, P]_{FN}$  is a version of the Riemann curvature (see [5], page 78). Finally, given a Lie algebroid  $\mathcal{A}$  and  $N \in \Gamma(\mathcal{A}^* \otimes \mathcal{A})$  such that  $[N, N]_{FN} = 0$ , one can construct a new (deformed) Lie algebroid  $\mathcal{A}_N$  (cf. [3, 6]). Moreover, Frölicher-Nijenhuis calculus is useful in geometric mechanics where it allows to give an intrinsic formulation of Euler-Lagrange equations. In this field, Lie algebroids have also been shown to be a useful tool to deal with systems with some kinds of symmetries.

In [8], P. Michor obtained a short expression for the Frölicher-Nijenhuis bracket on manifolds in terms of the covariant Lie derivatives. A formula for the Frölicher-Nijenhuis bracket on Lie algebroids in supergeometric language was obtained by P. Antunes in [1]. In this paper we define some operators relevant for Frölicher-Nijenhuis calculus in the setting of Lie algebroids, including the covariant Lie derivative, and study their properties. In this way we are able to extend Michor's formula for Frölicher-Nijenhuis bracket to Lie algebroids.

## 2. COVARIANT LIE DERIVATIVE ON LIE ALGEBROIDS

Let  $(\mathcal{A}, [\ , \ ], \rho)$  be a Lie algebroid over a manifold  $M$ , and  $E$  a vector bundle over  $M$ . We write  $\Omega^k(\mathcal{A}, E) = \Gamma(\wedge^k \mathcal{A}^* \otimes E)$  for the space of skew-symmetric  $E$ -valued  $k$ -forms on  $\mathcal{A}$ . If  $E = M \times \mathbb{R}$  is the trivial line bundle over  $M$ , we denote  $\Omega^k(\mathcal{A}, E)$  by  $\Omega^k(\mathcal{A})$ .

We write  $\Sigma_m$  for the permutation group on  $\{1, \dots, m\}$ . For  $k$  and  $s$  such that  $k + s = m$ , we denote by  $\text{Sh}_{k,s}$  the subset of  $(k, s)$ -shuffles in  $\Sigma_m$ . Thus  $\sigma \in \text{Sh}_{k,s}$

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if and only if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+s).$$

Similarly, for a triple  $(k, l, s)$ , such that  $k + l + s = m$ , we denote by  $\text{Sh}_{k,l,s}$  the subset of  $(k, l, s)$ -shuffles in  $\Sigma_m$ , that is the set of permutations  $\sigma$ , such that

$$\begin{aligned} \sigma(1) < \sigma(2) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l), \\ \sigma(k+l+1) < \cdots < \sigma(k+l+s). \end{aligned}$$

For a  $k$ -form  $\omega \in \Omega^k(\mathcal{A})$  and  $\phi \in \Omega^p(\mathcal{A}, E)$ , we define the form  $\omega \overline{\wedge} \phi \in \Omega^{k+p}(\mathcal{A}, E)$  by

$$(\omega \overline{\wedge} \phi)(Z_1, \dots, Z_{p+k}) = \sum_{\sigma \in \text{Sh}_{k,p}} (-1)^\sigma \omega(Z_{\sigma(1)}, \dots, Z_{\sigma(k)}) \phi(Z_{\sigma(k+1)}, \dots, Z_{\sigma(k+p)}).$$

Here and everywhere in this paper  $Z_1, \dots, Z_{p+k}$  denote arbitrary sections of the Lie algebroid  $\mathcal{A}$ . If  $E = M \times \mathbb{R}$  is the trivial line bundle over  $M$ , we denote  $\overline{\wedge}$  by  $\wedge$ , and  $\Omega^*(\mathcal{A})$  becomes a commutative graded algebra with the multiplication given by  $\wedge$ . Further, note that  $\Omega^*(\mathcal{A}, E)$  is an  $\Omega^*(\mathcal{A})$ -module with the action given by  $\overline{\wedge}$ . For any  $\omega \in \Omega^k(\mathcal{A})$  we define the operator  $\epsilon_\omega$  on  $\Omega^*(\mathcal{A}, E)$  by

$$\begin{aligned} \epsilon_\omega : \Omega^*(\mathcal{A}, E) &\rightarrow \Omega^{*+k}(\mathcal{A}, E) \\ \phi &\mapsto \omega \overline{\wedge} \phi \end{aligned}$$

Sometimes, given a operator  $A$  we will use  $\omega \wedge A$  as an alternative notation for  $\epsilon_\omega A$ .

Let  $\phi \in \Omega^p(\mathcal{A}, \mathcal{A})$ . For any vector bundle  $E$  over  $M$ , we define the operator  $i_\phi$  on  $\Omega^*(\mathcal{A}, E)$  by

$$(1) \quad (i_\phi \psi)(Z_1, \dots, Z_{p+k}) = \sum_{\sigma \in \text{Sh}_{p,k}} (-1)^\sigma \psi(\phi(Z_{\sigma(1)}, \dots, Z_{\sigma(p)}), Z_{\sigma(p+1)}, \dots, Z_{\sigma(p+k)})$$

where  $\psi \in \Omega^{k+1}(\mathcal{A}, E)$ .

We say that  $\nabla : \Gamma(\mathcal{A}) \times \Gamma(E) \rightarrow \Gamma(E)$  is an  $\mathcal{A}$ -connection on  $E$  (see [7]) if

- 1)  $\nabla_X$  is an  $\mathbb{R}$ -linear endomorphism of  $\Gamma(E)$ ;
- 2)  $\nabla s$  is a  $\mathcal{C}^\infty(M)$ -linear map from  $\Gamma(\mathcal{A})$  to  $\Gamma(E)$ ;
- 3)  $\nabla_X(fs) = (\rho(X)f)s + f\nabla_X s$  for any  $f \in \mathcal{C}^\infty(M)$ ,  $X \in \Gamma(\mathcal{A})$ , and  $s \in \Gamma(E)$ .

The curvature of an  $\mathcal{A}$ -connection  $\nabla$  is defined by

$$R(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s.$$

It is easy to check that  $R$  is tensorial and skew-symmetric in the first two arguments, thus we can consider  $R$  as an element of  $\Omega^2(\mathcal{A}, \text{End}(E))$ , where  $\text{End}(E)$  is the endomorphism bundle of  $E$ .

Given an  $\mathcal{A}$ -connection on a vector bundle  $E$ , we define the covariant exterior derivative on  $\Omega^*(\mathcal{A}, E)$  by

$$\begin{aligned} (d^\nabla \phi)(Z_1, \dots, Z_{p+1}) &= \sum_{\sigma \in \text{Sh}_{1,p}} (-1)^\sigma \nabla_{Z_{\sigma(1)}}^E (\phi(Z_{\sigma(2)}, \dots, Z_{\sigma(p+1)})) \\ &\quad - \sum_{\sigma \in \text{Sh}_{2,p-1}} (-1)^\sigma \phi([Z_{\sigma(1)}, Z_{\sigma(2)}], Z_{\sigma(3)}, \dots, Z_{\sigma(p+1)}). \end{aligned}$$

Note that  $d^\nabla$  is related to the curvature  $R$  of  $\nabla^E$  by the formula

$$((d^\nabla)^2 \phi)(Z_1, \dots, Z_{p+2}) = \sum_{\sigma \in \text{Sh}_{2,p}} (-1)^\sigma R(Z_{\sigma(1)}, Z_{\sigma(2)}) (\phi(Z_{\sigma(3)}, \dots, Z_{\sigma(p+2)})).$$

**Definition 1.** A derivation of degree  $k$  on  $\Omega^*(\mathcal{A}, E)$  is a linear map  $D: \Omega^*(\mathcal{A}, E) \rightarrow \Omega^{*+k}(\mathcal{A}, E)$  such that

$$D(\omega \bar{\wedge} \phi) = \bar{D}(\omega) \bar{\wedge} \phi + (-1)^{kp} \omega \bar{\wedge} D(\phi)$$

for all  $\omega \in \Omega^p(\mathcal{A})$  and  $\phi \in \Omega^*(\mathcal{A}, E)$ , where  $\bar{D}: \Omega^*(\mathcal{A}) \rightarrow \Omega^*(\mathcal{A})$  is some map.

For any derivation  $D$  on  $\Omega^*(\mathcal{A}, E)$  and  $\alpha \in \Omega^*(\mathcal{A})$ , we have

$$[D, \epsilon_\alpha] = \epsilon_{\bar{D}\alpha}.$$

In particular, the map  $\bar{D}$  is unique for a given derivation  $D$  on  $\Omega^*(\mathcal{A}, E)$ . Let  $\omega_1 \in \Omega^{p_1}(\mathcal{A})$ ,  $\omega_2 \in \Omega^{p_2}(\mathcal{A})$ . From the following computation

$$\begin{aligned} D((\omega_1 \wedge \omega_2) \bar{\wedge} \phi) &= \bar{D}(\omega_1 \wedge \omega_2) \bar{\wedge} \phi + (-1)^{k(p_1+p_2)} \omega_1 \wedge \omega_2 \bar{\wedge} D(\phi) \\ D(\omega_1 \bar{\wedge} (\omega_2 \bar{\wedge} \phi)) &= \bar{D}(\omega_1) \wedge \omega_2 \bar{\wedge} \phi + (-1)^{kp_1} \omega_1 \bar{\wedge} D(\omega_2 \bar{\wedge} \phi) \\ &= \bar{D}(\omega_1) \wedge \omega_2 \bar{\wedge} \phi + (-1)^{kp_1} \omega_1 \wedge \bar{D}(\omega_2) \bar{\wedge} \phi + (-1)^{k(p_1+p_2)} \omega_1 \wedge \omega_2 \bar{\wedge} D(\phi) \end{aligned}$$

one can see that  $\bar{D}$  is a derivation on  $\Omega^*(\mathcal{A})$ .

It is easy to check that for any given  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$ ,  $i_\phi$  is a derivation of degree  $k-1$ , and  $d^\nabla$  is a derivation of degree 1 on  $\Omega^*(\mathcal{A}, E)$ . The *covariant Lie derivative*  $\mathcal{L}_\phi^\nabla$  is defined as the *graded commutator*  $[i_\phi, d^\nabla] = i_\phi d^\nabla + (-1)^k d^\nabla i_\phi$ . The graded commutator of two derivations of degree  $k$  and  $l$  is a derivation of degree  $k+l$ . In particular,  $\mathcal{L}_\phi^\nabla$  is a derivation of degree  $k$  for any  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$ .

Suppose we have an  $\mathcal{A}$ -connection  $\nabla$  on  $\mathcal{A}$ . We will say that  $\nabla$  is torsion-free if  $\nabla_X Y - \nabla_Y X = [X, Y]$  for all  $X, Y \in \Gamma(\mathcal{A})$ . On every algebroid  $(\mathcal{A}, [\cdot, \cdot], \rho)$ , there exists a torsion-free  $\mathcal{A}$ -connection. Namely, one can take an arbitrary bundle metric on  $\mathcal{A}$  and the associated Levi-Civita connection on  $\mathcal{A}$ . Given  $\mathcal{A}$ -connections  $\nabla^{\mathcal{A}}$  on  $\mathcal{A}$  and  $\nabla^E$  on  $E$ , we define  $\nabla_X s \in \Omega^p(\mathcal{A}, E)$  for every  $s \in \Omega^p(\mathcal{A}, E)$  by

$$(\nabla_X s)(Z_1, \dots, Z_p) := \nabla_X^E(s(Z_1, \dots, Z_p)) - \sum_{t=1}^p s(Z_1, \dots, \nabla_X^{\mathcal{A}} Z_t, \dots, Z_p).$$

It is easy to check that for any  $s \in \Omega^k(\mathcal{A}, E)$ ,  $X \in \Gamma(\mathcal{A})$ , and a torsion-free  $\mathcal{A}$ -connection on  $\mathcal{A}$ , we have  $\mathcal{L}_X^\nabla s = \nabla_X s + i_{\nabla X} s$  and  $\nabla X = d^\nabla X$ . In other words  $\nabla_X = \mathcal{L}_X^\nabla - i_{d^\nabla X}$ . Motivated by this relation, we define for  $\phi \in \Omega^p(\mathcal{A}, \mathcal{A})$  an operator  $\nabla_\phi$  on  $\Omega^*(\mathcal{A}, E)$  by

$$(2) \quad \nabla_\phi := \mathcal{L}_\phi^\nabla - (-1)^p i_{d^\nabla \phi}.$$

Note that  $\nabla_\phi$  depends on two connections: an  $\mathcal{A}$ -connection on  $E$  and a torsion-free  $\mathcal{A}$ -connection on  $\mathcal{A}$ . Since  $\nabla_\phi$  is a linear combination of two derivations of degree  $p$ , we see that  $\nabla_\phi$  is a derivation of degree  $p$ . The following proposition shows that for  $s \in \Omega^*(\mathcal{A}, E)$  the map  $\nabla s: \Omega^*(\mathcal{A}, \mathcal{A}) \rightarrow \Omega^*(\mathcal{A}, E)$  is a homomorphism of  $\Omega^*(\mathcal{A})$ -modules.

**Proposition 2.** For any  $\omega \in \Omega^p(\mathcal{A})$ ,  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$ , and  $s \in \Omega^*(\mathcal{A}, E)$ , we have

$$\nabla_{\omega \bar{\wedge} \phi} s = (\omega \wedge \nabla_\phi) s = \epsilon_\omega \nabla_\phi s = \omega \bar{\wedge} (\nabla_\phi s).$$

*Proof.* The equation

$$\mathcal{L}_{\omega \bar{\wedge} \phi}^\nabla = [i_{\omega \bar{\wedge} \phi}, d^\nabla] = [\omega \wedge i_\phi, d^\nabla] = (-1)^{k+p} (d\omega) \wedge i_\phi + \omega \wedge \mathcal{L}_\phi^\nabla$$

implies that  $\omega \wedge \mathcal{L}_\phi^\nabla = \mathcal{L}_{\omega \bar{\wedge} \phi}^\nabla - (-1)^{p+k} i_{(d\omega) \bar{\wedge} \phi}$ . Now we have

$$\begin{aligned} \omega \wedge \nabla_\phi s &= \omega \wedge \mathcal{L}_\phi^\nabla s - (-1)^p \omega \wedge i_{d^\nabla \phi} s = \mathcal{L}_{\omega \bar{\wedge} \phi}^\nabla s - (-1)^{p+k} i_{(d\omega) \bar{\wedge} \phi} s - (-1)^p i_{\omega \bar{\wedge} d^\nabla \phi} s \\ &= \mathcal{L}_{\omega \bar{\wedge} \phi}^\nabla s - (-1)^{p+k} i_{d\omega \bar{\wedge} \phi + (-1)^k \omega \bar{\wedge} d^\nabla \phi} s = \nabla_{\omega \bar{\wedge} \phi} s. \end{aligned}$$

□

It was proven in [10] that the commutator  $[i_\phi, i_\psi]$  for  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$  and  $\psi \in \Omega^l(\mathcal{A}, \mathcal{A})$  is given by the formula

$$(3) \quad [i_\phi, i_\psi] = i_{i_\phi \psi} - (-1)^{(k-1)(l-1)} i_{i_\psi \phi}.$$

**Theorem 3.** *Let  $\nabla$  be a torsion-free  $\mathcal{A}$ -connection on  $\mathcal{A}$  and  $\nabla^E$  be an  $\mathcal{A}$ -connection on a vector bundle  $E$ . For  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$  and  $\psi \in \Omega^l(\mathcal{A}, \mathcal{A})$  we have on  $\Omega^*(\mathcal{A}, E)$*

$$(4) \quad [\nabla_\phi, i_\psi] = i_{\nabla_\phi \psi} - (-1)^{k(l-1)} \nabla_{i_\psi \phi}.$$

*Proof.* First we check the claim for  $\phi = X \in \Gamma(\mathcal{A})$  and  $\psi = Y \in \Gamma(\mathcal{A})$ . Let  $s \in \Omega^{p+1}(\mathcal{A}, E)$ . We get

$$\begin{aligned} (\nabla_X i_Y s)(Z_1, \dots, Z_p) &= \nabla_X^E(s(Y, Z_1, \dots, Z_p)) - \sum_{t=1}^p s(Y, Z_1, \dots, \nabla_X Z_t, \dots, Z_p) \\ &= (\nabla_X s)(Y, Z_1, \dots, Z_p) + s(\nabla_X Y, Z_1, \dots, Z_p) \\ &= (i_Y \nabla_X s)(Z_1, \dots, Z_p) + (i_{\nabla_X Y} s)(Z_1, \dots, Z_p). \end{aligned}$$

Thus  $[\nabla_X, i_Y] = i_{\nabla_X Y}$ . Since (4) is additive in  $\phi$  and  $\psi$ , it is enough to prove it for  $\phi = \alpha \bar{\wedge} X$ ,  $\psi = \beta \bar{\wedge} Y$ , where  $\alpha \in \Omega^k(\mathcal{A})$ ,  $\beta \in \Omega^l(\mathcal{A})$ , and  $X, Y \in \Gamma(\mathcal{A})$ . Repeatedly using Proposition 2 and  $[\nabla_X, i_Y] = i_{\nabla_X Y}$ , we get

$$\begin{aligned} [\nabla_{\alpha \bar{\wedge} X}, i_{\beta \bar{\wedge} Y}] &= [\alpha \wedge \nabla_X, \beta \wedge i_Y] = [\epsilon_\alpha, \beta \wedge i_Y] \nabla_X + \epsilon_\alpha [\nabla_X, \beta \wedge i_Y] \\ &= (-1)^{kl} \epsilon_\beta [\epsilon_\alpha, i_Y] \nabla_X + \epsilon_\alpha [\nabla_X, \epsilon_\beta] i_Y + \epsilon_\alpha \epsilon_\beta [\nabla_X, i_Y] \\ &= -(-1)^{kl-l} \epsilon_\beta \epsilon_{i_Y \alpha} \nabla_X + \epsilon_\alpha \epsilon_{\nabla_X \beta} i_Y + \epsilon_\alpha \epsilon_\beta i_{\nabla_X Y} \\ &= i_{\alpha \wedge \nabla_X \beta \bar{\wedge} Y + \alpha \wedge \beta \bar{\wedge} \nabla_X Y} + (-1)^{(k-1)l} \nabla_{\beta \wedge i_Y \alpha \bar{\wedge} X} \\ &= i_{\alpha \bar{\wedge} \nabla_X (\beta \bar{\wedge} Y)} + (-1)^{(k-1)l} \nabla_{\beta \wedge i_Y (\alpha \bar{\wedge} X)} \\ &= i_{\nabla_{\alpha \bar{\wedge} X} (\beta \bar{\wedge} Y)} + (-1)^{(k-1)l} \nabla_{i_{\beta \bar{\wedge} Y} (\alpha \bar{\wedge} X)}. \end{aligned}$$

□

To formulate the next result, we extend the definition of  $R$  by defining for any  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$  and  $\psi \in \Omega^l(\mathcal{A}, \mathcal{A})$  the form  $R(\phi, \psi) \in \Omega^{k+l+1}(\mathcal{A}, \mathcal{A})$  as follows

$$\begin{aligned} R(\phi, \psi)(Y_1, \dots, Y_{k+l+1}) &= \\ &= \sum_{\sigma \in \text{Sh}_{k,l,1}} R(\phi(Y_{\sigma(1)}, \dots, Y_{\sigma(p)}), \psi(Y_{\sigma(p+1)}, \dots, Y_{\sigma(p+q)})) Y_{\sigma(p+q+1)}. \end{aligned}$$

**Theorem 4.** *Let  $\nabla$  be a torsion-free  $\mathcal{A}$ -connection on  $\mathcal{A}$  and  $\nabla^E$  a flat  $\mathcal{A}$ -connection on a vector bundle  $E$  over  $M$  (i.e.  $\nabla^E$  is a representation of  $\mathcal{A}$ ). Then for any  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$ ,  $\psi \in \Omega^l(\mathcal{A}, \mathcal{A})$ , we have the following equality on  $\Omega^*(\mathcal{A}, E)$*

$$(5) \quad [\nabla_\phi, \nabla_\psi] = \nabla_{\nabla_\phi \psi} - (-1)^{kl} \nabla_{\nabla_\psi \phi} - i_{R(\phi, \psi)}.$$

*Proof.* First we prove (5) for  $\phi = X, \psi = Y \in \Gamma(\mathcal{A})$ . For  $s \in \Omega^p(\mathcal{A})$ , we get

$$\begin{aligned} (\nabla_X \nabla_Y s)(Z_1, \dots, Z_p) &= \nabla_X^E(\nabla_Y^E s(Z_1, \dots, Z_p)) - \sum_{s=1}^p \nabla_Y^E s(Z_1, \dots, \nabla_X Z_s, \dots, Z_p) \\ &= \nabla_X^E \nabla_Y^E s(Z_1, \dots, Z_p) - \sum_{s=1}^p \nabla_X^E s(Z_1, \dots, \nabla_Y Z_s, \dots, Z_p) \\ &\quad - \sum_{s=1}^p \nabla_Y^E s(Z_1, \dots, \nabla_X Z_s, \dots, Z_p) + \sum_{s=1}^p s(Z_1, \dots, \nabla_Y \nabla_X Z_s, \dots, Z_p) \\ &\quad + \sum_{s \neq t} s(Z_1, \dots, \nabla_Y Z_t, \dots, \nabla_X Z_s, \dots, Z_p). \end{aligned}$$

By anti-symmetrization of the above formula in  $X$  and  $Y$  and using that  $\nabla^E$  is flat, we get

$$[\nabla_X, \nabla_Y]s(Z_1, \dots, Z_p) = \nabla_{[X, Y]}^E(s(Z_1, \dots, Z_p)) - \sum_{s=1}^p s(Z_1, \dots, [\nabla_X, \nabla_Y]Z_s, \dots, Z_p).$$

Further

$$\begin{aligned} (\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X})s(Z_1, \dots, Z_p) &= \nabla_{\nabla_X Y - \nabla_Y X}^E(s(Z_1, \dots, Z_p)) \\ &\quad - \sum_{s=1}^p s(Z_1, \dots, (\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X})Z_s, \dots, Z_p). \end{aligned}$$

Taking the difference of the last two formulas and using the definition of  $R$  and that  $\nabla$  torsion-free, we have

$$(([\nabla_X, \nabla_Y] - \nabla_{\nabla_X Y} + \nabla_{\nabla_Y X})s)(Z_1, \dots, Z_p) = (-i_{R(X, Y)}s)(Z_1, \dots, Z_p).$$

Since (5) is additive in  $\phi$  and  $\psi$ , it is enough to prove it for  $\phi = \alpha \bar{\wedge} X$  and  $\psi = \beta \bar{\wedge} Y$ , where  $\alpha \in \Omega^k(\mathcal{A})$ ,  $\beta \in \Omega^l(\mathcal{A})$ , and  $X, Y \in \Gamma(\mathcal{A})$ . Using the already proved case and Proposition 2, we get

$$\begin{aligned} [\nabla_{\alpha \bar{\wedge} X}, \nabla_{\beta \bar{\wedge} Y}] &= [\alpha \wedge \nabla_X, \beta \wedge \nabla_Y] = [\epsilon_\alpha, \beta \wedge \nabla_Y] \nabla_X + \epsilon_\alpha [\nabla_X, \beta \wedge \nabla_Y] \\ &= (-1)^{kl} \epsilon_\beta [\epsilon_\alpha, \nabla_Y] \nabla_X + \epsilon_\alpha [\nabla_X, \epsilon_\beta] \nabla_Y + \epsilon_\alpha \epsilon_\beta [\nabla_X, \nabla_Y] \\ &= -(-1)^{kl} \epsilon_\beta \nabla_Y \alpha \nabla_X + \epsilon_\alpha \nabla_X \beta \nabla_Y + \epsilon_\alpha \epsilon_\beta (\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X} - i_{R(X, Y)}). \end{aligned}$$

Repeatedly using Proposition 2, we see that  $[\nabla_{\alpha \bar{\wedge} X}, \nabla_{\beta \bar{\wedge} Y}]$  can be written as  $\nabla_\theta + i_\tau$ , where

$$\begin{aligned} \theta &= -(-1)^{kl} \beta \wedge \nabla_Y \alpha \bar{\wedge} X + \alpha \wedge \nabla_X \beta \bar{\wedge} Y + \alpha \wedge \beta \bar{\wedge} \nabla_X Y - \alpha \wedge \beta \bar{\wedge} \nabla_Y X \\ &= \alpha \bar{\wedge} \nabla_X (\beta \bar{\wedge} Y) - (-1)^{kl} (\beta \bar{\wedge} \nabla_Y (\alpha \bar{\wedge} X)) = \nabla_\phi \psi - (-1)^{kl} \nabla_\psi \phi \end{aligned}$$

and

$$\tau = -\alpha \wedge \beta \bar{\wedge} R(X, Y) = -R(\alpha \bar{\wedge} X, \beta \bar{\wedge} Y) = -R(\phi, \psi).$$

This finishes the proof.  $\square$

Note that the connection  $\nabla_X^\rho f := \rho(X)f$  defined on the trivial line bundle  $M \times \mathbb{R} \rightarrow M$  is obviously flat. Thus (5) holds on  $\Omega^*(\mathcal{A})$ , if  $\nabla$  is defined via  $\nabla^\rho$  and any torsion-free connection on  $\mathcal{A}$ .

### 3. THE FRÖLICHER-NIJENHUIS BRACKET ON LIE ALGEBROIDS

In [10], Nijenhuis defined the Frölicher-Nijenhuis bracket on Lie algebroids of  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$  and  $\psi \in \Omega^l(\mathcal{A}, \mathcal{A})$  by an equality of operators on  $\Omega^*(\mathcal{A})$  equivalent to

$$(6) \quad [\mathcal{L}_\phi^\nabla, i_\psi] = i_{[\phi, \psi]_{FN}} - (-1)^{k(l-1)} \mathcal{L}_{i_\psi \phi}^\nabla.$$

He also obtained a formula for computing  $[\phi, \psi]_{FN}$ . In the next theorem we give an alternative formula using the covariant Lie derivatives, which extends the one obtained in [8] to the Lie algebroids setting.

**Theorem 5.** *Let  $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$  and  $\psi \in \Omega^l(\mathcal{A}, \mathcal{A})$ . Suppose  $\nabla$  be a torsion-free  $\mathcal{A}$ -connection on  $\mathcal{A}$ . Then*

$$[\phi, \psi]_{FN} = \mathcal{L}_\phi^\nabla \psi - (-1)^{kl} \mathcal{L}_\psi^\nabla \phi.$$

*Proof.* By (2) we have

$$[\mathcal{L}_\phi^\nabla, i_\psi] = [\nabla_\phi + (-1)^k i_{d^\nabla \phi}, i_\psi] = [\nabla_\phi, i_\psi] + (-1)^k [i_{d^\nabla \phi}, i_\psi].$$

Hence, using (3) and (4) we get

$$[\mathcal{L}_\phi^\nabla, i_\psi] = i_{\nabla_\phi \psi} - (-1)^{k(l-1)} \nabla_{i_\psi \phi} + (-1)^k i_{i_{d^\nabla \phi} \psi} - (-1)^{kl} i_{i_\psi d^\nabla \phi}.$$

Next, using (2) in the second summand we have

$$\begin{aligned} [\mathcal{L}_\phi^\nabla, i_\psi] &= -(-1)^{k(l-1)} \left( \mathcal{L}_{i_\psi \phi}^\nabla - (-1)^{k+l-1} i_{d^\nabla i_\psi \phi} \right) \\ &\quad + i_{\nabla_\phi \psi} + (-1)^k i_{i_{d^\nabla \phi} \psi} - (-1)^{kl} i_{i_\psi d^\nabla \phi}. \end{aligned}$$

Notice that the subscripts of  $\mathcal{L}^\nabla$  in (6) and in the above formula are the same. Hence, due to the injectivity of  $\phi \mapsto i_\phi$ , we get by comparing the subscripts of  $i$  that

$$\begin{aligned} [\phi, \psi]_{FN} &= (-1)^{k(l-1)} (-1)^{k+l-1} d^\nabla i_\psi \phi + \nabla_\phi \psi + (-1)^k i_{d^\nabla \phi} \psi - (-1)^{kl} i_\psi d^\nabla \phi \\ &= \nabla_\phi \psi + (-1)^k i_{d^\nabla \phi} \psi - (-1)^{kl} (i_\psi d^\nabla \phi - (-1)^{l-1} d^\nabla i_\psi \phi) \end{aligned}$$

Finally, using the definitions of  $\nabla_\phi$  and of  $\mathcal{L}_\psi^\nabla$  we get the claimed result.  $\square$

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